Low-energy representation of the projected BCS Hamiltonian close to half filling

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We investigate a connection between the *t-J* model and the strongly correlated Bardeen-Cooper-Schrieffer (BCS) Hamiltonian, with the effect of strong electron correlations accounted by the Gutzwiller projection. We show that in the immediate vicinity of half filling the projected two-dimensional BCS Hamiltonian with strong pairing develops an antiferromagnetically ordered ground state. This result explicitly demonstrates that antiferromagnetism in this model appears as a natural consequence of the strong Coulomb repulsion in a low doped regime. At moderate doping the ground state of the Gutzwiller-projected BCS Hamiltonian is, in finite-system studies, known to become qualitatively similar to Anderson's resonating valence bond state which, in turn, fits nicely with the properties of the *t-J* model in that regime. Combined together, these two properties indicate that the projected BCS Hamiltonian at least qualitatively captures the essential low-energy physics of the *t-J* model in the underdoped region.

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I. INTRODUCTION

The investigation of strongly correlated electron systems has been a central issue in solid-state physics for more than four decades. The discovery of high-T_c superconductivity in copper-oxide based compounds (cuprates) revived the interest in simple models displaying such strong correlations. Two much investigated models are the Hubbard model and its "descendant," the *t-J* model.[1,2] One of the main theoretical questions in that field is whether or not there is a superconducting phase in the *t-J* model.[1] Besides, the interplay between antiferromagnetism (AF) and superconductivity in the cuprates as well as their sensitivity to doping is still not very well understood.

It is clear that, in superconducting state induced by electron-electron interaction, the formation of Cooper pairs must also reflect strong electron correlations. As a result the Bardeen-Cooper-Schrieffer (BCS) effective Hamiltonian should be directly modified by the inclusion of a nondouble occupancy (NDO) constraint to account for such an effect.

In a recent paper, Park[3] discussed a close connection between the *t-J* model and the Gutzwiller-projected BCS Hamiltonian. It was shown both numerically and analytically that the ground states of the *t-J* model at half filling [i.e., of the two-dimensional (2D) antiferromagnetic Heisenberg model] and of the strongly correlated BCS Hamiltonian are equivalent to each other. Moreover, at sufficiently small doping, there is numerical evidence of a strong overlap between those two ground-state wave functions, which provides further support for the existence of superconductivity in the *t-J* model. Clearly it would be interesting to establish by analytical means such an equivalence at nonzero hole concentration. As is well known, slightly away from half filling the long-range AF order is still observed in the cuprate superconductors. If the projected BCS Hamiltonian is indeed believed to contain, close to half filling, the low-energy physics of the *t-J* Hamiltonian, its ground state must also exhibit the AF order in the immediate vicinity of half filling. This manifests itself as a quite nontrivial necessary condition for the low-energy physics described by the Gutzwiller-projected BCS Hamiltonian to be considered identical to that of the *t-J* Hamiltonian at sufficiently low doping.

The purpose of the present report is to investigate the Gutzwiller-projected BCS Hamiltonian analytically, close to half filling. We do not address here the issue of the properties of the *t-J* model at moderate doping, concentrating our full attention instead to the region of the phase diagram very close to half filling. We derive the low-energy long-wavelength effective action for the lightly doped 2D projected BCS Hamiltonian on a bipartite lattice. The action obtained is shown to be identical to that of the 2D quantum antiferromagnetic Heisenberg model explicitly represented by the three-dimensional (3D) nonlinear σ model. In other words, close to half filling, the ground state of the Gutzwiller-projected BCS Hamiltonian is antiferromagnetically ordered and nonsuperconducting. Since the conventional BCS Hamiltonian does not exhibit any magnetic ordering and always displays superconductivity, those results explicitly demonstrate that antiferromagnetism appears as a natural consequence of the strong Coulomb repulsion at low doped regimes.

Formally, the Gutzwiller projection takes care of the strong electron correlation due to the large on-site Coulomb repulsion. Close to half filling, the strong short-range Coulomb repulsion between lattice electrons brings about the superexchange in the emergent local spin moments by means of the virtual exchange processes which involve virtual creation of the electron-spin singlets on the nearest-neighbor (nn) empty sites. As a result the Gutzwiller-projected strong-pairing BCS Hamiltonian can be described in terms of the emergent spin-spin exchange interaction and the AF ordering continues from half filling up to a small doping. In other words the projected BCS Hamiltonian can describe both the superconducting state at moderate doping and, in contrast

with the conventional BCS Hamiltonian, the ordered magnetic phase for sufficiently low doping.

This paper is organized as follows. In Sec. II we set up the necessary notation and emphasize the importance of the Gutzwiller projection close to half filling. In Sec. III the low-energy action for the Gutzwiller-projected BCS Hamiltonian is derived within the coherent-state path-integral approach. We provide an independent operator derivation in Sec. IV. A discussion of our results is presented in Sec. V. We conclude in Sec. VI by discussing some physical implications of the obtained low-energy representation of the projected BCS Hamiltonian. Some necessary technical details are listed in Appendixes A–C.

II. GUTZWILLER-PROJECTED BCS HAMILTONIAN

We start with the Gutzwiller-projected BCS Hamiltonian on a 2D bipartite lattice, $L=A\oplus B$:

$$H_{\mathrm{BCS}}^{G}=\hat{\mathcal{P}}_{G}H_{\mathrm{BCS}}\hat{\mathcal{P}}_{G}=\hat{\mathcal{P}}_{G}(H_{t}+H_{\Delta})\hat{\mathcal{P}}_{G}, \tag{1}$$

where

$$H_{t}=-t\sum_{ij\sigma}(c_{i\sigma}^{\dagger}c_{j\sigma}+\mathrm{H.c.})$$

is a kinetic term of strength t responsible for the hopping of electrons from one lattice site to its nearest neighbor and

$$H_{\Delta}=\sum_{ij}\Delta_{ij}(c_{i\uparrow}^{\dagger}c_{j\downarrow}^{\dagger}-c_{i\downarrow}^{\dagger}c_{j\uparrow}^{\dagger}+\mathrm{H.c.}) \tag{2}$$

is the pairing term in real space. Here $c_{i\sigma}$ is the electron annihilation operator at site i with the spin projection $\sigma=\uparrow\downarrow$.

At every lattice site the Gutzwiller projection operator,

$$\hat{\mathcal{P}}_{G}=\prod_{i}(1-n_{i\sigma}n_{i-\sigma}),\quad n_{i\sigma}=c_{i\sigma}^{\dagger}c_{i\sigma},$$

projects out the doubly occupied states $|\uparrow\downarrow\rangle$ thereby reducing the quantum Hilbert space to a lattice site product of the three-dimensional spaces, $\mathcal{H}_{\mathrm{pr}}$, spanned by $|0\rangle_{i}$, $|\uparrow\rangle_{i}$, and $|\downarrow\rangle_{i}$. Physically this modification of the original Hilbert space results in strong electron correlation effects which are believed to account for the unusual and rich physics of the high-T_{c} superconductors.

Upon introducing a full set of the on-site operators $X^{ab}:=|a\rangle\langle b|$, $a,b=0,\uparrow,\downarrow$, which are also referred to as the Hubbard operators, the Gutzwiller projection is explicitly evaluated to be

$$\hat{\mathcal{P}}_{G}c_{i\sigma}^{\dagger}\hat{\mathcal{P}}_{G}=c_{i\sigma}^{\dagger}(1-n_{i-\sigma})=X_{i}^{\sigma 0},$$

$$\hat{\mathcal{P}}_{G}n_{i}\hat{\mathcal{P}}_{G}\equiv\tilde{n}_{i}=n_{i}-2n_{i\uparrow}n_{i\downarrow}=X_{i}^{\uparrow\uparrow}+X_{i}^{\downarrow\downarrow},$$

where $X^{\uparrow\uparrow}+X^{\downarrow\downarrow}+X^{00}=|0\rangle\langle 0|+|\uparrow\rangle\langle\uparrow|+|\downarrow\rangle\langle\downarrow|=\hat{I}$ is the identity operator in the on-site Gutzwiller-projected Hilbert space. Note that the eigenvalues of the projected electron number operator, $\tilde{n}_{i}$, are either zero or one so that the doubly occupied states are prohibited. It should be stressed that, close to half filling, the Gutzwiller projection is of a crucial importance: the projected electron operator $\hat{\mathcal{P}}_{G}c_{i\sigma}^{\dagger}\hat{\mathcal{P}}_{G}$ in this region significantly differs from the bare electron operator $c_{i\sigma}^{\dagger}$ (right at half filling $\hat{\mathcal{P}}_{G}c_{i\sigma}^{\dagger}\hat{\mathcal{P}}_{G}=0$).

With these notations Eq. (1) can be rewritten in the equivalent form,

$$H_{\mathrm{BCS}}^{G}=-t\sum_{ij\sigma}(X_{i}^{\sigma 0}X_{j}^{0\sigma}+\mathrm{H.c.})+\tilde{\mu}_{h}\sum_{i}X_{i}^{00}+\sum_{ij}\Delta_{ij}(X_{i}^{\uparrow 0}X_{j}^{\downarrow 0}-X_{i}^{\downarrow 0}X_{j}^{\uparrow 0}+\mathrm{H.c.}), \tag{3}$$

where we have introduced the chemical-potential term to control the total number of doped holes, $N_{h}=\Sigma_{i}X_{i}^{00}$, where

$$X_{i}^{00}=\hat{\mathcal{P}}_{G}(1-n_{i})\hat{\mathcal{P}}_{G}=1-n_{\uparrow i}-n_{\downarrow i}+n_{\uparrow i}n_{\downarrow i}.$$

The local NDO constraint is rigorously taken into account at the expense of the introduction of the Hubbard operators with more complicated commutation relations than those of the standard fermion algebra. In fact, fermionic operators $X_{i}^{\sigma 0}$ together with the bosonic ones $X_{i}^{\sigma\sigma'}$ form, on every lattice site, a basis of the fundamental representation of the graded Lie algebra SU(2|1) given by the (anti)commutation relations

$$\{X_{i}^{ab},X_{j}^{cd}\}_{\pm}=(X_{i}^{ad}\delta^{bc}\pm X_{j}^{bc}\delta^{ad})\delta_{ij}, \tag{4}$$

where the (+) sign should be used only when both operators are fermionic.

In the strong-pairing limit ($|\Delta|\gg t$) the projected BCS Hamiltonian (3) reduces to

$$H_{\Delta}^{G}=\sum_{ij}\Delta_{ij}(X_{i}^{\uparrow 0}X_{j}^{\downarrow 0}-X_{i}^{\downarrow 0}X_{j}^{\uparrow 0}+\mathrm{H.c.})+\tilde{\mu}_{h}\sum_{i}X_{i}^{00}. \tag{5}$$

In contrast with the conventional real-space BCS Hamiltonian, the strongly correlated BCS Hamiltonian given by Eq. (5) is not an exactly solvable model. Since the Hubbard operators appear as the elements of the SU(2|1) superalgebra, a natural framework to address this problem is provided by the SU(2|1) coherent-state path-integral representation of the partition function.

Note finally that the chemical potential $\tilde{\mu}_{h}$ we deal with in this paper is served to control a number of the doped carriers in the *projected* Hilbert space. In contrast, the chemical potential μ_{e} commonly used in the unprojected BCS Hamiltonian,

$$H_{\mathrm{BCS}}=H_{t}+H_{\Delta}+\mu_{e}\sum_{i\sigma}n_{i\sigma},$$

fixes a total number of the electrons in the full Hilbert space where the on-site double electron occupancy is allowed. It should be kept in mind that those two potentials exhibit, as functions of doping, quite different properties. We explicitly elaborate on this point in Sec. V. Throughout this paper we consistently use the chemical potential $\tilde{\mu}_{h}$ which from now on is denoted as $\mu:=\tilde{\mu}_{h}$.

III. SU(2|1) COHERENT-STATE PATH INTEGRAL REPRESENTATION OF THE PARTITION FUNCTION

In the SU(2|1) coherent-state basis the partition function,

$$Z_\Delta = tr\ \exp(-\beta H_\Delta^G),$$

takes the form of the SU(2|1) coherent-state phase-space path integral (see Appendix B):

$$Z_\Delta = \int D\mu(z,\xi) e^{S_\Delta}, \tag{6}$$

where

$$D\mu(z,\xi) = \prod_{i,t} \frac{d\bar{z}_i(t) dz_i(t)}{2\pi i (1+|z_i|^2)^2} d\bar{\xi}_i(t) d\xi_i(t).$$

Here z_i is a complex number that keeps track of the spin degrees of freedom while ξ_i is a complex Grassmann parameter that describes the charge degrees of freedom.

The effective action,

$$S_\Delta = i\sum_i \int_0^\beta a_i(t)dt - \sum_i \int_0^\beta \bar{\xi}_i(\partial_t + ia_i)\xi_i dt - \int_0^\beta H_\Delta^{G,cl} dt, \tag{7}$$

involves the U(1)-valued connection one-form of the magnetic monopole bundle (see Appendix A) that can formally be interpreted as a spin "kinetic" term,

$$ia = -\langle z|\partial_t|z\rangle = \frac{1}{2}\frac{\dot{\bar{z}}z - \bar{z}\dot{z}}{1+|z|^2},$$

with $|z\rangle$ being the SU(2) coherent state. This term is also frequently referred to as the Berry connection. The dynamical part of the action takes the form

$$H_\Delta^{G,cl} = \sum_{ij}\left(\Delta_{ij}\xi_i\xi_j \frac{\bar{z}_j - \bar{z}_i}{\sqrt{(1+|z_i|^2)(1+|z_j|^2)}} + \text{H.c.}\right) + \mu\sum_i \bar{\xi}_i\xi_i. \tag{8}$$

Here $z_i(t)$ and $\xi_i(t)$ are the dynamical fields. This representation rigorously incorporates the constraint of no double occupancy. Since the NDO constraint is explicitly resolved in representation (6), the dynamical variables z_i and ξ_i bear no local gauge redundancy associated with the constraint-generated local gauge transformations, and in contrast with the slave-particle fields, are gauge independent. Note also that the on-site representation (6) and (7) at $H_\Delta = 0$ yields $Z_{H_\Delta=0} = dim\mathcal{H}_{\text{pr}} = 3$, as it should.[4]

Under the global SU(2) rotation,

$$z_i \to \frac{uz_i + v}{-\bar{v}z_i + \bar{u}}, \quad \xi_i(t) \to e^{i\psi_i}\xi_i, \quad a_i \to a_i + d\psi_i, \tag{9}$$

where

$$\psi_i = -i\log\sqrt{\frac{-v\bar{z}_i + u}{-\bar{v}z_i + \bar{u}}}, \quad \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in \text{SU}(2). \tag{10}$$

Effective action (7) is invariant under the global spin rotations given by Eq. (9).

Let us now make the following change in variables on the sublattice B,

$$z_i \to -\frac{1}{\bar{z}_i}, \quad \xi_i \to \bar{\xi}_i\sqrt{\frac{z_i}{\bar{z}_i}}, \quad i \in B. \tag{11}$$

This transformation is equivalent to SU(2) rotation (9) with $u=0$ and $v=1$ followed by a complex conjugation. Under this transformation, $\vec{S}_i \to -\vec{S}_i$ and the gauge potential a_i changes its sign, $a_i \to -a_i$. The effective action then becomes

$$\begin{aligned} S_\Delta \to S_\Delta = i\sum_{i\in A}\int_0^\beta a_i(t)dt - \mu N\beta/2 \\ + \sum_{i\in A}\int_0^\beta \bar{\xi}_i(-\partial_t - ia_i - \mu)\xi_i dt \\ + \sum_{i\in B}\int_0^\beta \bar{\xi}_i(-\partial_t - ia_i + \mu)\xi_i dt - \int_0^\beta H_\Delta^{G,cl} dt, \end{aligned}$$

where

$$H_\Delta^{G,cl} = \sum_{ij}(\Delta_{ij}\bar{\xi}_i\xi_j\langle z_j|z_i\rangle + \text{H.c.}),$$

and $\langle z_i|z_j\rangle$ stands for an inner product of the SU(2) coherent states,

$$\langle z_i|z_j\rangle = \frac{1+\bar{z}_i z_j}{\sqrt{(1+|z_j|^2)(1+|z_i|^2)}}. \tag{12}$$

This can be written in the form

$$S_\Delta = i\sum_{i\in A}\int_0^\beta a_i(t)dt - \mu N\beta/2 + \sum_{ij}\int_0^\beta \bar{\xi}_i(t)G_{ij}^{-1}(t,s)\xi_j(s)dtds, \tag{13}$$

where

$$G_{ij}^{-1}(t,s) = G_{(0)ij}^{-1}(t,s) - ia_i(t)\delta_{ij}\delta(t-s) + \Sigma_{ij}(t)\delta(t-s),$$

with $\Sigma_{ij} = \Delta_{ij}\langle z_j|z_i\rangle$ and

$$G_{(0)ij}^{-1}(t,s) = \delta_{ij}(-\partial_t - \mu)\delta(t-s), \quad i \in A,$$

$$G_{(0)ij}^{-1}(t,s) = \delta_{ij}(-\partial_t + \mu)\delta(t-s), \quad i \in B.$$

The fermionic degrees of freedom in Eq. (6) can formally be integrated out to yield

$$\begin{aligned} \int D\bar{\xi}D\xi \exp\left(\sum_{ij}\int_0^\beta \bar{\xi}_i(t)G_{ij}^{-1}(t,s)\xi_j(s)dtds\right) \\ = \exp\ Tr\ \log\ G^{-1} \\ = \exp[Tr\ \log\ G_{(0)}^{-1} + Tr\ \log(1 - G_{(0)}ia + G_{(0)}\Sigma)]. \end{aligned} \tag{14}$$

Here the trace has to be carried out over both space and time indices. Calculating explicitly a factor that comes from the zero-order Green's function, we get

$$Z_0 := Z_{\Delta=a=0} = \exp(Tr\ \log\ G_{(0)}^{-1} - \mu N\beta/2)$$
$$= \exp\left(\beta\sum_{i\in A}\log\ G_{(0)}^{-1} + \beta\sum_{i\in B}\log\ G_{(0)}^{-1} - \mu N\beta/2\right)$$
$$= \left(2\cosh\frac{\mu\beta}{2}e^{-(\mu\beta/2)}\right)^N,$$

which is a correct result for the partition function of N noninteracting spinless fermions,

$$Z_0 = tr\ \exp\left(-\mu\int_0^\beta\sum_i f_i^\dagger f_i\right).$$

Up to this point no approximation has been made in the derivation of the effective action. In fact, we are interested in a derivation of an effective action to describe a low-energy dynamics of the spin degrees of freedom of the projected strong-pairing Hamiltonian close to half filling. For that purpose we deduce an effective action in the spin degrees of freedom by performing a perturbative expansion of the expression $Tr\log(1-G_{(0)}ia+G_{(0)}\Sigma)$ in powers of $|\Delta|/\mu \ll 1$. Physically, this corresponds to the lightly doped region of the phase diagram (see Sec. V). The second step consists of expanding the obtained representation up to first order in ∂_t and second order in Δ_{ij} implying that eventually we will set $i \to j$. This amounts to the so-called gradient expansion that corresponds to the low-energy and long-wavelength limit of the action. In this way we obtain

$$Tr\ \log(1 - G_{(0)}ia + G_{(0)}\Sigma) = -Tr(G_{(0)}ia) - \frac{1}{2}Tr(G_{(0)}\Sigma G_{(0)}\Sigma). \tag{15}$$

Note that $Tr(G_{(0)}iaG_{(0)}\Sigma)=0$ since $\Sigma_{ii}=0$. This expansion is justified in the limit $|\Delta|/\mu \ll 1$, $\mu\beta \gg 1$.

The a-dependent term in Eq. (15) contributes to the action in the following way

$$-Tr(G_{(0)}ia) = -i\sum_{i\in A}G_{(0)i}(0^-)\int_0^\beta a_i(t)dt$$
$$-i\sum_{i\in B}G_{(0)i}(0^-)\int_0^\beta a_i(t)dt,$$

where $G_{(0)i}(0^-) := \lim_{\epsilon\to 0}G_{(0)i}(-\epsilon)$, $\epsilon > 0$, and

$$G_{(0)i}^{A/B}(\tau) = \frac{e^{\mp\mu\tau}}{1+e^{\pm\mu\beta}} - \theta(\tau)e^{\mp\mu\tau}. \tag{16}$$

Here the upper sign corresponds to the case $i\in A$, whereas the lower one corresponds to the case $i\in B$. Explicit representation (16) tells us that

$$\Delta S_1 := -Tr(G_{(0)}ia) = -i\sum_{i\in B}\int_0^\beta a_i(t)dt + \mathcal{O}(e^{-\mu\beta}), \tag{17}$$

where it is implied that $\mu\beta \gg 1$.

Let us now turn to the second term in Eq. (15). We get

$$\Delta S_2 := -\frac{1}{2}Tr(G_{(0)}\Sigma G_{(0)}\Sigma)$$
$$= -\frac{1}{2}\sum_{ij}\int G_{(0)i}(t_1-t_2)\Sigma_{ij}(t_2)G_{(0)j}(t_2-t_1)\Sigma_{ji}(t_1)dt_1dt_2.$$

Introducing new variables, $\tau=\frac{t_1-t_2}{2}$, $\eta=\frac{t_1+t_2}{2}$, and expanding the product $\Sigma_{ij}(\eta+\tau)\Sigma_{ji}(\eta-\tau)=\Sigma_{ij}(\eta)\Sigma_{ji}(\eta)+\mathcal{O}(\tau)$ (this corresponds to the gradient expansion in imaginary time[5]) gives to the lowest order

$$\Delta S_2 = -\frac{1}{2}\sum_{ij}\int_{-\beta}^\beta G_{(0)}^A(\tau)G_{(0)}^B(-\tau)d\tau\int_0^\beta\Sigma_{ij}(\eta)\Sigma_{ji}(\eta)d\eta. \tag{18}$$

With the help of Eq. (16) we get

$$-\frac{1}{2}\int_{-\beta}^\beta G_{(0)}^A(\tau)G_{(0)}^B(-\tau)d\tau = \frac{1}{2\mu}[1+\mathcal{O}(e^{-\mu\beta})]. \tag{19}$$

The effective spin action is given by the sum of all the term evaluated above:

$$Z_\Delta^{\text{eff}}/Z_0 = \int D\mu(z,\bar{z})e^{S_\Delta^{\text{eff}}}, \tag{20}$$

where the SU(2) invariant measure factor

$$D\mu(z,\bar{z}) = \prod_{i,t}\frac{d\bar{z}_i(t)dz_i(t)}{2\pi i(1+|z_i|^2)^2},$$

and

$$S_\Delta^{\text{eff}} = i\sum_{i\in A}\int_0^\beta a_i(t)dt - i\sum_{i\in B}\int_0^\beta a_i(t)dt$$
$$+\sum_{ij}\int_0^\beta\frac{|\Delta_{ij}|^2}{2\mu}|\langle z_i|z_j\rangle|^2dt. \tag{21}$$

Let us now rotate the spin on the sublattice B back to their initial position, $z_i\to -1/\bar{z}_i$. Under this transformation

$$|\langle z_i|z_j\rangle|^2 \to 1-|\langle z_i|z_j\rangle|^2, \quad a_{i\in B}\to -a_{i\in B}.$$

In this way we finally get

$$S_\Delta^{\text{eff}} = i\sum_i\int_0^\beta a_i(t)dt - \sum_{ij}J_{ij}^{(U=\infty,\mu)}\int_0^\beta(|\langle z_i|z_j\rangle|^2-1)dt, \tag{22}$$

where the long-wavelength limit ($j\to i$) is implied. This action describes the antiferromagnetic Heisenberg model with the effective coupling (see Appendix A)

$$J_{ij}^{(U=\infty,\mu)} = |\Delta_{ij}|^2/2\mu > 0. \tag{23}$$

The resulting model is the low-energy action for the fully projected ($U=\infty$) BCS Hamiltonian in the vicinity of half filling (μ is large but finite). In the explicit low-energy and long-wavelength limit, 2D quantum action (22) reduces to that of the 3D classical nonlinear sigma model (see Appendix C). Taking into consideration the renormalization-group

analysis of that sigma model action,[6] it then follows that the ground state of the 2D Gutzwiller-projected BCS Hamiltonian is AF ordered at sufficiently low doping.

Right at half filling $\mu \to \infty$, producing in this case $J_{ij}^{(U=\infty,\mu)} \to 0$. However, this does not contradict Park's observation that the projected BCS Hamiltonian (for large but finite U) possesses a long-range AF ordered ground state right at half filling.[7] To rederive Park's result one should back off from the infinite U limit. This can be seen as follows. Consider the Hamiltonian

$$H_{\Delta+U} = \sum_{ij} \Delta_{ij}(c_{i\uparrow}^{\dagger} c_{j\downarrow}^{\dagger} - c_{i\downarrow}^{\dagger} c_{j\uparrow}^{\dagger} + \text{H.c.}) + U \sum_{i} n_{i\uparrow} n_{i\downarrow}, \quad (24)$$

which, in the $U \to \infty$ limit, reduces to H_{Δ}^{G} as in Eq. (5). We now back off from the infinite U limit, the effects of double occupancy need to be built perturbatively in powers of Δ/U. We thus consider the partially projected BCS Hamiltonian.

Let us make the following unitary transformation of the electron operators for all sites $j \in B$:

$$c_{j\uparrow} \to c_{j\downarrow}^{\dagger}, \quad c_{j\downarrow} \to -c_{j\uparrow}^{\dagger}.$$

In this way we get

$$H_{\Delta+U} \to -\sum_{ij} \Delta_{ij}(c_{i\sigma}^{\dagger} c_{j\sigma} + \text{H.c.}) + U \sum_{i} n_{i\uparrow} n_{i\downarrow} - U \sum_{i \in B} n_i. \quad (25)$$

Using the representation

$$n_{i\uparrow} n_{i\downarrow} = -\frac{2}{3} \vec{Q}_i^2 + \frac{n_i}{2},$$

where $\vec{Q}_i$ is the electron-spin operator, we can write down the partition function as

$$Z_{\Delta+U} = \int D\vec{\phi} D\bar{\Psi} D\Psi \exp \int_0^{\beta} \mathcal{L}_{\Delta+U} dt, \quad (26)$$

with

$$\mathcal{L}_{\Delta+U} = \frac{-3U}{8} \sum_{i} \vec{\phi}_i^2 - U \sum_{i} \bar{\Psi}_i \vec{\tau} \vec{\phi}_i \Psi_i + \sum_{ij} \Delta_{ij}(\bar{\Psi}_i \Psi_j + \text{H.c.}) + U \sum_{i \in B} \bar{\Psi}_i \Psi_i,$$

$$\Psi_i = (c_{i\uparrow}, c_{i\downarrow})^t.$$

For $U \gg \Delta$ one gets $\vec{\phi}_i^2 \approx 1$. As a result, in this limit, one can make the identification $\vec{\phi}_i = 2\vec{S}_i^{cl}(\bar{z}, z)$.[8] We use the identity

$$2\vec{S}^{cl}\vec{\tau} = V \tau_z V^{\dagger},$$

where

$$V = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix}. \quad (27)$$

Moreover rotating now the spinors to the z axis, $\Psi \to V\Psi$, we get

$$\mathcal{L}_{\Delta+U} \to \sum_{i} \bar{\Psi}_i(-\partial_t - U\tau_z)\Psi_i + U \sum_{i \in B} \bar{\Psi}_i \Psi_i + \sum_{i} \bar{\Psi}_i V_i^{\dagger}(-\partial_t V_j)\Psi_j + \sum_{ij} \Delta_{ij} \bar{\Psi}_i V_i^{\dagger} V_j \Psi_j.$$

The fermionic degrees of freedom can now be integrated out in the low-energy limit, yielding

$$Z_{\Delta+U}/Z_0 = \int D\mu(z,\bar{z}) e^{S_{\Delta+U}^{\text{eff}}}, \quad (28)$$

where

$$D\mu(z,\bar{z}) = \prod_{i,t} \frac{d\bar{z}_i(t) dz_i(t)}{2\pi i (1+|z_i|^2)^2},$$

and the effective low-energy action is again given by Eq. (22) but now with $J_{ij} = |\Delta_{ij}|^2/2U > 0$. This consideration provides an independent proof of the equivalence of the low-energy physics of the 2D Heisenberg AF model and the Gutzwiller-projected strong-pairing Hamiltonian at half filling first established in Ref. 3 within a different approach.

IV. SECOND-ORDER OPERATOR PERTURBATION THEORY

In this section we briefly comment on another derivation of the low-energy representation of the projected BCS Hamiltonian now following a more conventional operator approach. Let us rewrite the Hamiltonian given in Eq. (5) in the following way:

$$H_{\Delta}^{G} = H_0 + V, \quad (29)$$

with

$$H_0 = \mu \sum_{i} X_i^{00}, \quad (30)$$

$$V = \sum_{ij} \Delta_{ij}(X_i^{\uparrow 0} X_j^{\downarrow 0} - X_i^{\downarrow 0} X_j^{\uparrow 0} + \text{H.c.}). \quad (31)$$

At half filling, we take $\mu \to \infty$. This ensures that any state with a finite number of holes is projected out from the theory. Close to half filling, μ is large and therefore we can treat V as a perturbation to H_0. The ground state of H_0 contains no holes and is highly degenerate, corresponding to all possible spin orientations in the half-filled limit. We denote this manifold by $|0_g\rangle$.

Let us now define the operator P_0 that projects into the subspace with no holes, that is, it projects into the ground state of H_0. Up to second order in V, we can define the effective Hamiltonian:[9]

$$H_{\text{eff}}^{\text{gr}} = P_0 V P_0 + \sum_{\phi_n \neq O_g} \frac{P_0 V |\phi_n\rangle\langle\phi_n| V P_0}{\epsilon_0 - \epsilon_n}, \quad (32)$$

where $\epsilon_0 = 0$ is the ground-state energy and $|\phi_n\rangle$ is an eigenstate of H_0 with eigenvalue ϵ_n.

Since V does not conserve the number of holes, the first-order contribution is zero, that is, $P_0 V P_0 = 0$. In the next or-

der, we have to calculate matrix elements such as $\langle \phi_n | V | 0_g \rangle$. They are nonzero only if $|\phi_n\rangle$ is a state containing two holes. In other words, the second-order term is related to virtual transitions where two neighboring holes are first created and then destroyed, that is, Cooper pairing fluctuations in the system. It is therefore clear that $\epsilon_0 - \epsilon_n = -2\mu$ and that our effective Hamiltonian now becomes

$$H_{\text{eff}}^{\text{gr}} = -\sum_{ij} \frac{|\Delta_{ij}|^2}{2\mu} (X_i^{\uparrow 0} X_j^{\downarrow 0} - X_i^{\downarrow 0} X_j^{\uparrow 0}) \times (X_j^{0\downarrow} X_i^{0\uparrow} - X_j^{0\uparrow} X_i^{0\downarrow}). \tag{33}$$

At half filling,

$$X_i^{\uparrow\downarrow} = S_i^+, \quad X_i^{\downarrow\uparrow} = S_i^-, \tag{34}$$

$$X_i^{\uparrow\uparrow} - X_i^{\downarrow\downarrow} = 2S_i^z, \quad X_i^{\uparrow\uparrow} + X_i^{\downarrow\downarrow} = \tilde{n}_i = 1, \tag{35}$$

and after a straightforward algebra we get

$$H_{\text{eff}}^{\text{gr}} = \sum_{ij} \frac{|\Delta_{ij}|^2}{\mu} \left(\vec{S}_i \vec{S}_j - \frac{1}{4} \right). \tag{36}$$

As a result, using a simple perturbative scheme, we find that the ground-state of the strong-pairing Gutzwiller-projected BCS Hamiltonian is indeed identical to that of the antiferromagnetic Heisenberg model with coupling given by $J_{ij} = \frac{|\Delta_{ij}|^2}{\mu}$, close to half filling.

V. DISCUSSION

Our results point out that the long-range AF order in the low-energy physics of the strongly correlated BCS Hamiltonian is obtained not only at half filling but also in its vicinity. Our derivation is strictly based on the analysis of the Gutzwiller-projected BCS Hamiltonian in the strong-pairing limit, i.e., $\Delta/t \to \infty$. However, slightly away from half filling there may in principle be some effects due to a finite hopping amplitude t. In general, the magnitude of t is quite large, which may raise some objections to our results obtained by having totally neglected the effects of the finite kinetic term.

In this section we show that those effects are in fact very small close to half filling and can therefore be safely discarded in this region. We start with the full Gutzwiller-projected BCS Hamiltonian including the hopping term, H_{BCS}^G, given by Eq. (3). The effective path-integral action is given now by Eq. (7) with an extra contribution, $-\int_0^\beta H_t^{cl}$, where

$$H_t^{cl} = -t \sum_{ij} (\xi_i \bar{\xi}_j \langle z_i | z_j \rangle + \text{H.c.}). \tag{37}$$

Here the overlap of the spin coherent states, $\langle z_i | z_j \rangle$, is given by Eq. (12). Right at half filling H_{BCS}^G develops the AF long-range ordered ground state. Let us recall now that we work on a 2D bipartite lattice with nn interactions only. The AF long-range order implies then that $\vec{S}_i^{cl} = -\vec{S}_j^{cl}$, where i and j denote the nn lattice sites. To enforce these constraints one must have $z_i = -1/\bar{z}_j$ [see Eq. (A3)]. From Eq. (12) it then follows immediately that $\langle z_i | z_j \rangle = 0$ so that the kinetic term makes no contribution to the total action. Very close to half filling (in the region where the AF long-range order still exists), one gets $\vec{Q}_i^{cl} = -\vec{Q}_j^{cl}$, where $\vec{Q}_i$ is the electron-spin operator (see Appendix B). In this case for a small enough hole concentration δ, one gets

$$z_i = -1/\bar{z}_j + \mathcal{O}(\delta).$$

which in turn implies that

$$\langle z_i | z_j \rangle = \mathcal{O}(\delta), \quad \delta \to 0. \tag{38}$$

All this indicates that, very close to half filling, the underlying ground-state AF background enforces the following dynamical renormalization of the hopping amplitude,

$$t \to \delta t.$$

Note that this type of the kinetic term renormalization has been used successfully in the "renormalized mean-field theory" of Zhang *et al.*[10] to treat the t-J model close to half filling. It is clear that in the vicinity of half filling the AF spin background strongly suppresses hopping of the doped holes around the lattice, reducing their contribution to the whole dynamics to a mere weakening of the AF order parameter. In contrast, if the spin background at half filling were instead ferromagnetic (FM), the whole situation would change drastically. In that case one would get $z_i \approx z_j$ and $\langle z_i | z_j \rangle \approx 1$. This would mean that the hole hopping would be greatly favored by the FM spin arrangement in such a way that even a small concentration of the doped holes could have changed the whole physical scenario altogether. This indeed happens in the so-called Nagaoka limit of the t-J model.[4]

Hamiltonian (3) may formally be related to the Gutzwiller-projected t-J Hamiltonian close to half filling with the J term treated in a mean-field level. Namely, the projected t-J model Hamiltonian in terms of the Hubbard operators reads,

$$H_{tJ} = -t \sum_{ij\sigma} (X_i^{\sigma 0} X_j^{0\sigma} + \text{H.c.}) + J \sum_{ij} \left(\vec{Q}_i \vec{Q}_j - \frac{\tilde{n}_i \tilde{n}_j}{4} \right), \tag{39}$$

where $\vec{Q}_i$ is the electron-spin operator [see Eq. (B2)]. Using the singlet-pair-creation operators $B_{ij}^\dagger = \frac{1}{2}(X_i^{\uparrow 0} X_j^{\downarrow 0} - X_i^{\downarrow 0} X_j^{\uparrow 0})$, the magnetic part of the t-J Hamiltonian can identically be rewritten in the form

$$H_J = -J \sum_{ij} B_{ij}^\dagger B_{ij}.$$

Close to half filling the hopping term, H_t, is effectively renormalized: $t \to \delta t$ with $\delta \ll 1$. If the hole concentration is small enough and J is fixed, one gets $J \gg t\delta$ so that H_t can be discarded. A conventional Hartree-Fock decoupling applied then to H_J yields

$$H_{tJ} = -t \sum_{ij\sigma} (X_i^{\sigma 0} X_j^{0\sigma} + \text{H.c.}) - \sum_{ij} (\Delta_{ij} B_{ij}^\dagger + \Delta_{ij}^* B_{ij} - |\Delta_{ij}|^2), \tag{40}$$

with J incorporated into Δ_{ij}. The parameter $\Delta_{ij} \neq 0$ indicates the onset of the electron-spin-singlet formation on the nn sites. Since $B_{ii} = 0$ there are no electron pairs occupying one

and the same lattice site. This is a manifestation of the strong electron correlation driven by the large on-site Coulomb repulsion. Unlike a conventional mean-field treatment, mean-field decoupling (40) is applied only after the local NDO constraint is rigorously imposed. One should keep in mind however that mean-field decomposition (40) cannot be rigorously justified and should at most be considered as a conjecture. Our results (and also those of Ref. 3) show that this conjecture is indeed highly plausible, at least close to half filling.

Another issue to be discussed further concerns the use of Δ/μ as a small parameter. For the unprojected BCS model at half filling, the chemical potential $\mu_e=0$. Close to half filling, μ_e should be small. As a result, Δ/μ_e does not represent a small parameter in this region. This is, however, not the case for the projected BCS Hamiltonian. To illustrate the difference we first consider the unprojected BCS Hamiltonian,

$$H_{\mathrm{BSC}} = -t\sum_{ij\sigma}(c_{i\sigma}^{+}c_{j\sigma}+\mathrm{H.c.}) - \mu_e\sum_{i\sigma}c_{i\sigma}^{\dagger}c_{i\sigma} + \Delta\sum_{ij}(c_{i\uparrow}c_{j\downarrow}-c_{i\downarrow}c_{j\uparrow}+\mathrm{H.c.}). \tag{41}$$

This Hamiltonian arises from a mean-field (MF) decoupling of the conventional Hamiltonian of the $t-J$ model in the particle-particle channel first proposed in Ref. 11. One obtains the following equation to determine the chemical potential at zero temperature

$$\frac{1}{N}\sum_{\vec{k}}\frac{t_{\vec{k}}-\mu_e}{E_{\vec{k}}} = \delta, \tag{42}$$

where $E_{\vec{k}}^2=(t_{\vec{k}}-\mu_e)^2+\Delta^2\gamma_{\vec{k}}^2$, $\gamma_{\vec{k}}=\cos k_x+\cos k_y$. It is clear that on a square lattice with the nn interaction one immediately gets $\delta=0$ at $\mu_e=0$. This can also be seen directly from representation (41): on a bipartite lattice with the nn interaction, this Hamiltonian at $\mu_e=0$ is invariant under a local unitary transformation of the fermionic operators,

$$c_{\uparrow}\rightarrow c_{\downarrow}^{\dagger}, \quad c_{\downarrow}\rightarrow -c_{\uparrow}^{\dagger}, \tag{43}$$

which implies $\langle n\rangle=2-\langle n\rangle$, or equivalently, $\delta=0$. In the strong-pairing limit, $\Delta\gg t$; from Eq. (42) it also follows that μ_e is small for small δ.

The situation however changes drastically if one deals with the fully projected ($U=\infty$) BCS Hamiltonian given by Eq. (3). To start with, substitution (43) does not represent a unitary transformation of the projected electron (Hubbard) operators. It cannot be used to fix a value of the hole concentration at $\mu=0$. Moreover, since the projected hole number operator, X^{00}, is a diagonal matrix with the eigenvalues zero and one, a chemical-potential term μX^{00} in Eq. (3) singles out, in the limit $\mu\rightarrow\infty$, the states with zero hole concentration.

To illustrate this, consider the exactly solvable BCS-type Hamiltonian for spinless fermions,

$$H = t\sum_{ij}f_i^{\dagger}f_j+\Delta\sum_{ij}(f_if_j+\mathrm{H.c.})+\mu\sum_i f_i^{\dagger}f_i. \tag{44}$$

This Hamiltonian arises from Eq. (40) in which one treats the spin content of the Hubbard operators at the MF level. In spite of its trivial form, this model can illustrate the importance of the local NDO constraint (which is enforced by definition since $f_i^2=0$) to establish the explicit relation between μ and $\delta=\langle f^{\dagger}f\rangle$ at zero temperature,

$$\frac{1}{2N}\sum_{\vec{k}}\frac{t_{\vec{k}}-\mu}{E_{\vec{k}}} = \delta-1/2, \tag{45}$$

where $t_{\vec{k}}=-2t\gamma_{\vec{k}}$, $E_{\vec{k}}^2=\Delta^2\beta_{\vec{k}}^2+(t_{\vec{k}}-\mu)^2$, and $\beta_{\vec{k}}=\sin k_x+\sin k_y$. In the case of strong pairing ($\Delta\gg t$) and close to half filling ($\delta\ll 1$), the above equation yields $(\Delta/\mu)^2\sim\delta+\mathcal{O}(\delta^2)$. This confirms that it is indeed meaningful to treat Δ/μ as a small parameter in the strong-pairing limit of the projected BCS Hamiltonian close to half filling. Physically, in this limit, there is no longer any Fermi surface and as a result the system turns into an insulator. This is a direct consequence of the strong electron correlations encoded in the NDO constraint.

Let us finally remark on the effect of the pairing symmetry of Δ_{ij}. We are primarily interested in a pairing with extended s-wave symmetry, where $\Delta_{ij}=\Delta$ for both $j=i+\hat{x}$ and $j=i+\hat{y}$, as well as in a pairing with the d-wave symmetry, where $\Delta_{ij}=\Delta$ for $j=i+\hat{x}$ and $\Delta_{ij}=-\Delta$ for $j=i+\hat{y}$. This symmetry is known to play no role exactly at half filling, where there is long-range AF order, since the saddle-point equations determining the AF order parameter are identical for both types of pairings.[3] Our representation (22) shows that this result continues to be true at very low but finite doping, where the AF order is still present. This happens because the effective AF coupling J_{ij} is proportional to $|\Delta_{ij}|^2$, and again the magnetically ordered phase of H_{BCS}^G does not distinguish between the s-wave and d-wave pairings.

VI. CONCLUSION

We conclude this work discussing the physical implications of the close connection between the Gutzwiller-projected BCS Hamiltonian and the t-J model. Our result shows that the ground state of the strong-pairing Gutzwiller-projected BCS Hamiltonian can indeed be considered as a reference state to a lightly doped Mott insulator that exhibits long-range AF order for small enough dopings. Technically, this is an important observation since it explicitly shows how an ordered magnetic state can evolve out of the strongly correlated slightly doped spin-liquid phase.

At moderate doping the spin-liquid state can be described by the Gutzwiller-projected BCS ground state, or in other words, by the short-range resonating valence bond (RVB) state proposed by Anderson.[12] Since the Gutzwiller projection does not commute with the BCS Hamiltonian,[3] this RVB state does not coincide with the ground state of the Gutzwiller-projected BCS Hamiltonian. In particular, the RVB state shows no long-range order even at half filling. In contrast, right at, as well as in the immediate vicinity of, half

filling the ground state of the strongly correlated BCS Hamiltonian exhibits long-range AF order as is observed in the cuprate superconductors.

On the other hand, the RVB state seems to be very successful in describing the properties of high T_c superconductors at moderate doping. In particular, at those dopings the RVB wave function and its improvements[13] yield good agreement with experiments[14] as well as with numerical studies on the t-J model.[15] As a result, that state is conjectured to be a good ansatz wave function for the t-J model in this region.[10] Hasegawa and Poilblanc[16] have shown that the overlap between the RVB state and the ground state of the t-J model is high ($\approx 90\%$) for the case of two holes in the $\sqrt{10}\times\sqrt{10}$ lattice system, approximately corresponding to a doping of $\delta=0.2$.

An interesting question then arises: what is the interrelationship between Anderson's RVB state and the ground state of the projected BCS Hamiltonian? It is quite clear that those states are qualitatively different right at and close to half filling. However, it seems quite plausible that those two states are very close to each other at moderate doping. At least there is a strong numerical evidence that for $\delta\approx 0.2$ both of these states fit very nicely with the properties of the t-J model. In particular, it has been shown that the optimal overlap at moderate doping between the ground state of the projected BCS Hamiltonian and that of the t-J model is more than 98% for the same $\sqrt{10}\times\sqrt{10}$ lattice cluster.[3] One may therefore conclude (based on such finite-system studies) that the ground-state wave function of the H^G_{BCS} Hamiltonian is a natural generalization of Anderson's RVB state to describe the t-J model in a region starting from a moderate doping down right to half filling.

However, we should stress that this close relationship between the RVB state and the ground state of the projected BCS Hamiltonian at moderate doping has not yet been established analytically. Note also that the low-energy action that corresponds to the strong-pairing projected BCS Hamiltonian, H^G_Δ, cannot in itself account for the weakening as well as for the eventual disappearance of the magnetic ordering when the hole concentration increases. This effect is produced by the growing influence of the kinetic t term that gradually destroys the long-range ordered state. Therefore, one needs to include the kinetic t term into consideration to regain the low-energy action that corresponds to the full Gutzwiller-projected BCS Hamiltonian. This takes us away from the strong-pairing limit discussed in the present work.

The strong-pairing limit is also no longer valid for moderate doping. In this case a numerical study of the Gutzwiller-projected BCS Hamiltonian can be used. Practically, to carry out numerical calculations one can employ the recently proposed doped-carrier representation of the projected electron operators which turns out to be very efficient close to, as well as moderately away from, half filling.[17,18] Specifically, the Gutzwiller-projected BCS Hamiltonian is much more amenable to the ansatz wave-function numerical studies than the t-J model Hamiltonian. As for a possible analytical study of the H^G_{BCS} Hamiltonian for small and moderate dopings, Abrikosov's diagrammatic technique in the doped-carrier representation can also be applied. Within that approach the infinitely large Lagrange multiplier that effectively enforces the NDO constraint can be incorporated in the fermionic dopon propagator as discussed in Ref. 18. In this way one arrives at a BCS-type theory of conventional fermions coupled to a compact U(1) lattice gauge field. The fermions with a modified propagator describe doped carriers (holes), whereas the gauge bosons represent the lattice spin degrees of freedom. The details will be reported elsewhere.

In summary, we believe that the Gutzwiller-projected BCS Hamiltonian encompasses at least qualitatively in a simple and physically appealing way the properties of the t-J model in the whole underdoped region. The crucial point is that the long-range AF order is effectively built into the ground state of the H^G_{BCS} Hamiltonian so that the AF order evolves directly out of the spin-liquid state in the limit of very small doping. However, only numerical evidences are at the present available for a close relationship between the two models at moderate doping. To address this issue theoretically more elaborated efforts are needed to open up a new perspective in analyzing the properties of high T_c superconductors in the whole underdoped regime.

APPENDIX A: SU(2) ALGEBRA AND COHERENT STATES

Consider the SU(2) algebra in the lowest $s=1/2$ representation:

$$[S_z,S_\pm]=\pm S_\pm,\quad [S_+,S_-]=2S_z,\quad \vec{S}^2=3/4. \tag{A1}$$

Acting with the "lowering" spin operator S^- on the "highest weight" state $|\uparrow\rangle$ we get the normalized SU(2) coherent state (CS) parametrized by a complex number z:

$$|z\rangle=\frac{1}{\sqrt{1+|z|^2}}\exp(zS^-)|\uparrow\rangle=\frac{1}{\sqrt{1+|z|^2}}(|\uparrow\rangle+z|\downarrow\rangle). \tag{A2}$$

In the basis spanned by the vectors $|\uparrow\rangle$ and $|\downarrow\rangle$, we have $S_+=|\uparrow\rangle\langle\downarrow|$, $S_-=|\downarrow\rangle\langle\uparrow|$, and $S_z=\frac{1}{2}(|\uparrow\rangle\langle\uparrow|-|\downarrow\rangle\langle\downarrow|)$. The CS symbols of the SU(2) generators are then easily evaluated to be ($S^{cl}:=\langle z|S|z\rangle$):

$$S^{cl}_+:=\frac{z}{1+|z|^2},\quad S^{cl}_-=\frac{\bar{z}}{1+|z|^2},$$

$$S^{cl}_z=\frac{1}{2}\frac{1-|z|^2}{1+|z|^2},\quad \vec{S}^2_{cl}=1/4,\quad (\vec{S}^2)_{cl}=3/4. \tag{A3}$$

There is a one-to-one correspondence between SU(2) generator (A1) and their CS symbols given by Eqs. (A3). Given a quantum Hamiltonian $H=H(\vec{S})$, the corresponding imaginary time phase-space action takes on the form,

$$\mathcal{A}_{\mathrm{SU(2)}}(\bar{z},z)=-\int_0^\beta \langle z|\frac{d}{dt}+H|z\rangle dt, \tag{A4}$$

with the kinetic term being given by

$$ia=-\langle z|\frac{d}{dt}|z\rangle=\frac{1}{2}\frac{\dot{\bar{z}}z-\bar{z}\dot{z}}{1+|z|^2}.$$

In particular, for the quantum $s=1/2$ Heisenberg model,

$$H = J\sum_{ij} (\vec{S}_i\vec{S}_j - 1/4),$$

one gets

$$H^{cl} = \frac{J}{2}\sum_{ij} (|\langle z_i|z_j\rangle|^2 - 1).$$

From the geometrical viewpoint, the SU(2) coherent states $|z\rangle$ can be thought of as sections of the magnetic monopole bundle $P[S^2, U(1)]$, with the U(1) connection one-form, ia, frequently refereed to as the Berry connection. Base space of that bundle, two-sphere S^2, appears as a classical phase space of spin, whereas its covariantly constant sections, $|z\rangle : (\partial_t + ia)|z\rangle = 0$, form a Hilbert space of a quantum spin.

APPENDIX B: SU(2|1) SUPERALGEBRA AND COHERENT STATES

Acting with the "lowering" superspin operators $X^{\downarrow\uparrow}$ and $X^{\downarrow 0}$ on the "highest weight" state $|\uparrow\rangle$, we get the normalized SU(2|1) coherent state in the 3D fundamental representation,

$$|z,\xi\rangle = (1 + \bar{z}z + \bar{\xi}\xi)^{-1/2}\exp(zX^{\downarrow\uparrow} + \xi X^{0\uparrow})|\uparrow\rangle$$
$$= (1 + \bar{z}z + \bar{\xi}\xi)^{-1/2}(|\uparrow\rangle + z|\downarrow\rangle + \xi|0\rangle), \quad \text{(B1)}$$

where z is a complex number, and ξ is a complex Grassmann parameter. The Grassmann parameter appears here due to the fact that $X^{\downarrow 0}$ is a fermionic operator in contrast with the operator $X^{\downarrow\uparrow}$. The product $\xi X^{0\uparrow}$ represents therefore a bosonic quantity as required.

At $\xi = 0$, the SU(2|1) CS reduces to the ordinary SU(2) CS, $|z, \xi = 0\rangle \equiv |z\rangle$ (A2), parametrized by a complex coordinate $z \in \mathrm{CP}^1 \simeq S^2$. In contrast, at $z = 0$, it represents a pure fermionic CS.

The CS symbols of the X operators, $X_{cl} := \langle z,\xi|X|z,\xi\rangle$, are

$$X^{0\downarrow}_{cl} = -\frac{z\bar{\xi}}{1+|z|^2}, \quad X^{\downarrow 0}_{cl} = -\frac{\bar{z}\xi}{1+|z|^2},$$

$$X^{0\uparrow}_{cl} = -\frac{\bar{\xi}}{1+|z|^2}, \quad X^{\uparrow 0}_{cl} = -\frac{\xi}{1+|z|^2},$$

$$Q^{+}_{cl} = X^{\uparrow\downarrow}_{cl} = \frac{z}{1+|z|^2}\left(1 - \frac{\bar{\xi}\xi}{1+|z|^2}\right),$$

$$Q^{-}_{cl} = X^{\downarrow\uparrow}_{cl} = \frac{\bar{z}}{1+|z|^2}\left(1 - \frac{\bar{\xi}\xi}{1+|z|^2}\right),$$

$$Q^{z}_{cl} = \frac{1}{2}(X^{\uparrow\uparrow}_{cl} - X^{\downarrow\downarrow}_{cl}) = \frac{1}{2}\frac{1-|z|^2}{1+|z|^2}\left(1 - \frac{\bar{\xi}\xi}{1+|z|^2}\right). \quad \text{(B2)}$$

Given a Hamiltonian as a polynomial function of the Hubbard operators, $H = H(X)$, the corresponding imaginary time phase-space action takes on the form,

$$\mathcal{A}_{SU(2|1)} = -\int_0^\beta \langle z,\xi|\frac{d}{dt} + H(X)|z,\xi\rangle dt, \quad \text{(B3)}$$

with the kinetic term given by

$$\langle z,\xi|\left(-\frac{d}{dt}\right)|z,\xi\rangle = \frac{1}{2}\frac{\dot{\bar{z}}z - \bar{z}\dot{z} + \dot{\bar{\xi}}\xi - \bar{\xi}\dot{\xi}}{1+|z|^2+\bar{\xi}\xi}. \quad \text{(B4)}$$

Substituting $H(X) = H_\Delta$ into Eq. (B3) and making the change in variables $z_i \to z_i\xi_i \to \xi_i\sqrt{1+|z_i|^2}$, we are led to effective action (7).

APPENDIX C: NONLINEAR σ MODEL

Consider the one-dimensional (1D) s-spin quantum AF Heisenberg model on a bipartite lattice, $L = A \oplus B$,

$$H_{AF} = \sum_{\langle ij\rangle} J_{ij}(\vec{S}_i\vec{S}_j - s^2), \quad J_{ij} > 0, \quad \text{(C1)}$$

where $J_{ij} = J$ for the nn sites and $J_{ij} = 0$ otherwise. Let us make the change $J \to J/2s$ and consider $H^{cl} = 2sH^{cl}_{s=1/2}$. The coherent-state action turns out to be proportional to $2s$ so that $S_{AF} = (2s)S_{s=1/2}$, where

$$H^{cl}_{s=1/2} = \sum_{\langle ij\rangle} \frac{J_{ij}}{2}(|\langle z_i|z_j\rangle|^2 - 1), \quad \text{(C2)}$$

so that

$$S_{s=1/2} = i\sum_{i\in} \int_0^\beta a_i(t)dt - \int_0^\beta H^{cl}_{s=1/2}$$

coincides with the action given by Eq. (22), provided we identity J_{ij} in Eq. (C2) with $|\Delta_{ij}|^2/\mu$.

To proceed, notice the following identity

$$|\langle z_i|z_j\rangle|^2 = \exp\Phi_{ij}, \quad \text{(C3)}$$

where

$$\Phi_{ij} = F(\bar{z}_i, z_j) + F(\bar{z}_j, z_i) - F(\bar{z}_i, z_i) - F(\bar{z}_j, z_j) \le 0, \quad \text{(C4)}$$

and $F(\bar{z}_i, z_j) = \log(1 + \bar{z}_i z_j)$ is the so-called SU(2) Kaehler potential, in terms of which the σ-model action can be derived.

In order to obtain the Néel ground state, we should have $\vec{S}^{cl}_A = -\vec{S}^{cl}_B$. To this end, let us make the following change in variables in the path integral:[19]

$$z_i \to z_i + \xi_i, \quad i \in A; \quad z_i \to -1/(\bar{z}_i - \bar{\xi}_i) \quad i \in B,$$

where $\xi_i, \bar{\xi}_i$ stand for a set of auxiliary fields $\sim a$. In this way we get

$$H^{cl}_{s=1/2} = \frac{J}{2}\sum_i F_{\bar{z}_i z_i}\frac{\partial\bar{z}}{\partial x_i}\frac{\partial z}{\partial x_i}a^2 + 2J\sum_i F_{\bar{z}_i z_i}\bar{\xi}_i\xi_i, \quad \text{(C5)}$$

where we have put $z_i = z(x_i)$, $z_{i+1} = z(x_i + a)$, with a being a lattice spacing. The total action becomes,

$$S_{s=1/2} = S_B + \int dt \sum_i F_{\bar{z}_i z_i} \left(\xi_i \dot{\bar{z}}_i - \bar{\xi}_i \dot{z}_i - 2J \bar{\xi}_i \xi_i - \frac{J}{2} \frac{\partial \bar{z}}{\partial x_i} \frac{\partial z}{\partial x_i} a^2 \right),$$

where S_B is the Berry phase term which will be considered shortly. The auxiliary fields $\bar{\xi}_i$ and ξ_i can be eliminated to yield

$$S_{s=1/2} = \int dt \sum_i F_{\bar{z}_i z_i} \left\{ -\frac{J}{2} \frac{\partial \bar{z}}{\partial x_i} \frac{\partial z}{\partial x_i} a^2 - \frac{1}{2J} \dot{\bar{z}}_i \dot{z}_i \right\} + S_B.$$

Restoring an explicit s dependence and going over to the continuum $(a \to 0)$ limit, finally yields

$$S_{\mathrm{AF}} = -\frac{1}{g^2} \int dx dt F_{\bar{z}z} (c \partial_x \bar{z} \partial_x z + c^{-1} \dot{\bar{z}} \dot{z}) + S_B, \qquad \text{(C6)}$$

where $c = 2Jsa$ is the spin-wave velocity, $g^2 = 1/s$ is the coupling of the σ model. We are free to choose units so that $c = 1$ and the action becomes Lorentz invariant:

$$S_{\mathrm{AF}} = -\frac{1}{g^2} \int dx dt (g_{\bar{z}z} \partial_\mu \bar{z} \partial_\mu z) = -\frac{1}{g^2} \int dx dt \frac{\partial_\mu \bar{z} \partial_\mu z}{(1 + |z|^2)^2},$$
$$\mu = 0, 1. \qquad \text{(C7)}$$

The generalization of this result to the case of the D-dimensional quantum antiferromagnet is trivial: in the low-energy quasiclassical (large spin s) limit, it is described by the classical D+1 dimensional σ model (C7) where $\mu = 0, 1, 2.., \mathrm{D}$.

The Berry phase term in 1D becomes

$$S_B = \frac{i}{2} \int_{S^2} da = s \int_{S^2} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} = (2\pi i) sN,$$

where N is an integer, the Brouwer degree of the map $z(x,t): S^2 \to S^2$. Thus, in 1D this phase term turns into a topological (metric independent) invariant that gives rise to dramatic consequences on ground-state degeneracy and low-energy spectrum. In higher dimensions the Berry phase term does not contribute to the action.

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